1. PHOTON ROCKET (5 points) — *Solution by Taavet Kalda, grading schemes by Oskar Vallhagen and Konstantin Dukach.*

i) *(1 point)* At non-relativistic speeds, we can apply classical momentum and energy conservation to find the acceleration in terms of the antimatter burning rate μ. In a time interval Δt, a mass of $\Delta m = \mu \Delta t$ antimatter annihilates with an equal mass of matter. The resulting photons have an energy equal to the annihilated rest energy $\Delta E = 2\Delta m c^2$. For maximal acceleration, the photons have to be all emitted in the same direction (can be achieved, for example, using mirrors). The resulting photon cloud will then have a momentum of $\Delta p = \Delta E/c$. From the conservation of momentum, the space ship must get a momentum boost in the opposite direction, equal to $\Delta p = M\Delta v = Mg\Delta t$. Combining everything, we get $\mu = Mg/(2c) = 1.635 \times 10^{-3}\,\text{kg/s}$.

Grading:

- Introducing time Δt and expressing $\Delta m = \mu \delta t$ **0.2 pts**.
- Photon energy $\Delta E = 2\Delta m c^2$ **0.2 pts**.
- Photon momentum $\Delta p = \Delta E/c$ **0.2 pts**.
- Rocket momentum $\Delta p = Mg\Delta t$ (half point if only force $F = Mg$ is stated) **0.2 pts**.
- Final answer **0.2 pts**.
- Missing factor 2 accounting for both antimatter and matter fom interstellar plasma **−0.2 pts**.
- Finding photon momentum by reasoning that the annihilated mass can be regarded as expelled with the speed of light **−0.2 pts**

ii) *(3 points)* The final speed is easiest to find by applying energy and momentum (4-momentum) conservation in the initial and final configurations. If the final rest mass of the ship is m_f, then $M - m_f$ of antimatter is burnt throughout the acceleration and an equal amount of matter is burnt from the interstellar space. Hence, it makes sense to consider the system of the ship + the burnt interstellar gas.

The initial rest mass of the system is $M + (M - m_f) = 2M - m_f = M_i$. This corresponds to an energy of $E_i = M_i c^2$ and momentum $p_i = 0$ due to the gas and the ship initially being at rest.

In the final configuration, we have the space ship moving with a speed of v with energy E_f and momentum p_f. We also have a photon cloud with energy E_p and momentum p_p, moving opposite to the direction of the space ship. Then the energy and momentum conservation simply read as $E_f + E_p = E_i$ and $p_i = 0 = p_f + p_p$. We also have the 4-momentum invariant for both the space ship and the photon gas. They read $E_f^2 - p_f^2 c^2 = m_f^2 c^4$ and $E_p^2 - p_p^2 c^2 = 0$. Solving the 4 equations (with 4 unknowns), we get

$$p_f = \frac{M_i^2 - m_f^2}{2M_i}c, \quad E_f = \frac{M_i^2 + m_f^2}{2M_i}c.$$

We could find the velocity by solving for v_f in the expression for the relativistic energy $E_f = m_f \gamma_f c^2$ with $\gamma_f = (1 - v_f^2/c^2)^{-0.5}$. A faster way, however, would be to use the expression for final momentum, $p_f = m_f \gamma_f v$, and notice that

$$v_f = \frac{p_f}{E_f}c^2 = \frac{M_i^2 - m_f^2}{M_i^2 + m_f^2}c$$
$$= \frac{(2M - m_f)^2 - m_f^2}{(2M - m_f)^2 + m_f^2}c$$
$$= \frac{180}{181}c \approx 0.9945c.$$

Grading:

- Realising that the problem can be solved by considering conservation of momentum and energy between initial and final state only **0.5 pts**.
- Correct conservation laws for momentum and energy (half of the points for each) **0.5 pts**.

Relating energy and momentum with 4-momentum invariants (or equivalent, half of the points each for rocket and photons) **1 pts**.
- Solving the equation system in terms of energies and momentum **0.5 pts**.
- Solving for v_f **0.5 pts**.

Non-relativistic solution:

- Realising that the problem can be solved by considering conservation of momentum and energy between initial and final state only **0.25 pts**.
- Correct conservation laws for momentum and energy (half of the points for each) **0.25 pts**.
- Relating energy and momentum to v_f **0.25 pts**.
- Final answer (if commenting on unphysical results, otherwise no points) **0.25 pts**.

Solution 2 by Oskar Vallhagen

An alternative way to solve the problem is to find an expression for the reactive force felt by the rocket, and use it to derive an equation for the resulting velocity change. During a time $\mathrm{d}t$, the energy released as photons from the burnt rocket antimatter is $\gamma \mu \mathrm{d}t c^2$ (including the kinetic energy of this mass) and the energy released from the ordinary matter needed for the annihilation is $\mu \mathrm{d}t c^2$. As the photon momentum scales as the inverse of the wavelength, which is red-shifted by a factor of $\sqrt{1 + v/c}/\sqrt{1 - v/c}$ in the rest frame compared to the rocket frame, the corresponding change in photon momentum in the rest frame is

$$\mathrm{d}p_{ph} = (\gamma + 1)\mu \mathrm{d}t c \sqrt{\frac{1 - v/c}{1 + v/c}}.$$

The force acting on the rocket can now be expressed as

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}v}(\gamma v)m\frac{\mathrm{d}v}{\mathrm{d}t} + \gamma v\frac{\mathrm{d}m}{\mathrm{d}t}$$
$$= m\left(\gamma^3\frac{v^2}{c^2} + \gamma\right)\frac{\mathrm{d}v}{\mathrm{d}t} - \gamma v\mu$$
$$= m\gamma^3\frac{\mathrm{d}v}{\mathrm{d}t} - \gamma v\mu = \frac{\mathrm{d}p_{ph}}{\mathrm{d}t}$$
$$= (\gamma + 1)\mu c\sqrt{\frac{1 - v/c}{1 + v/c}}.$$

We now change variable from time to mass by noting that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}m}\frac{\mathrm{d}m}{\mathrm{d}t} = -\frac{\mathrm{d}v}{\mathrm{d}m}\mu,$$

yielding

$$-m\gamma^3\frac{\mathrm{d}v}{\mathrm{d}m} - \gamma v = (\gamma + 1)c\sqrt{\frac{1 - v/c}{1 + v/c}}.$$

This equation can be solved by separation of variables according to

$$\int_M^{m_f}\frac{\mathrm{d}m}{m} = \int_0^{v_f} -\frac{\gamma^3}{\gamma v + (\gamma + 1)c\sqrt{\frac{1 - v/c}{1 + v/c}}}\mathrm{d}v,$$

which can be simplified to

$$\ln\left(\frac{m_f}{M}\right) = \int_0^{v_f/c} -\frac{\mathrm{d}x}{(1 - x)(\sqrt{1 - x^2} + x + 1)}$$
$$= I(v_f/c) - I(0),$$

with $x = v/c$ and I is the indefinite integral. To evaluate this integral, we first make the substitution $x = \sin y$. This gives $\sqrt{1 - x^2} = \cos y$ and $\mathrm{d}x = \cos y\,\mathrm{d}y$, so that the (indefinite) integral becomes

$$I = \int -\frac{\cos y\,\mathrm{d}y}{(1 - \sin y)(\cos y + \sin y + 1)}.$$

We now make the further substitution $z = \tan y/2$, which gives

$$\sin y = \frac{2z}{1 + z^2}, \quad \cos y = \frac{1 - z^2}{1 + z^2}, \quad \mathrm{d}y = \frac{2}{1 + z^2}\mathrm{d}z,$$

so that the integral becomes, after some manipulation,

$$I = \int -\frac{dz}{1-z} = \ln(1-z).$$

Substituting back, we use the trigonometric expressions $\sin y = 2\sin(y/2)\cos(y/2)$ and $\cos y = 2\cos^2(y/2) - 1$ to rewrite $z = \tan(y/2) = \sin(y/2)/\cos(y/2) = \sin y/(2\cos^2(y/2)) = \sin y/(1 + \cos y) = x/(1 + \sqrt{1-x^2})$, which after insertion into the original integral gives

$$\ln\left(\frac{m_f}{M}\right) = \ln\left(1 - \frac{x_f}{1 + \sqrt{1-x_f^2}}\right)$$

$$\Rightarrow \frac{1}{10} = 1 - \frac{x_f}{1 + \sqrt{1-x_f^2}}$$

$$\Rightarrow x_f = \frac{180}{181}.$$

Grading:

- Correct photon momentum change in the rest frame **0.5 pts**
- Expanding the momentum change of the rocket in terms of v and setting up total momentum (or force) balance **0.25 pts**
- Eliminate the time variable **0.25 pts**
- Express solution as an integral using separation of variables **0.5 pts**
- Simplifying the integral and first substitution **0.5 pts**
- Second substitution **0.5 pts**
- Substitute back and calculate final answer **0.5 pts**

Non-relativistic solution:

- Momentum balance **0.25 pts**
- Eliminate time variable and express as integral using separation of variables **0.25 pts**
- Evaluate the integral **0.25 pts**
- Final answer (if commenting on unphysical results, otherwise no points) **0.25 pts**

iii) *(1 point)* The last photon is emitted when the space ship moves at a speed of v_f. The

photon is observed by a stationary observer. We can directly use the relativistic Doppler shift effect to find

$$f_{\text{obs}} = f_0\sqrt{\frac{1-v/c}{1+v/c}} = \frac{m_f}{M_i}f_0$$

$$= \frac{m_f}{2M - m_f}f_0 = \frac{1}{19}f_0.$$

Grading:

- Correct relativistic expression for doppler shift (-0.3 pts if non-relativistic or only valid for $v_f \ll c$) **0.5 pts**
- Final answer (no points if unphysical, eg negative frequencies, even if consistent with previous calculations) **0.5 pts**

2. GAS AND FLUID FLOWS (10 points) — *Solution by Taavet Kalda, grading schemes by Joonas Kalda, Mihkel Kree, Andres Põldaru, Oleg Košik.*

i) *(1 point)* As the plate falls, it will rotate around the bump without slipping and push the air out from beneath it, at ever faster speeds, the closer it gets to the bottom plate. As such, part of the rotational energy of the plate is transferred over to the escaping air molecules. Further, the pressure and temperature of the gas is uniform, because of the incompressibility condition.

Since the problem is 2-dimensional, the mass, volumes, moment of inertia and other quantities are per unit length of the system (on the figure, into the page). Let x mark the distance from the bump and v denote the horizontal speed of air at x. There is a volume of air equal to $V(x) = xhx/(2L)$ between $x = 0$ and x. As the plate falls down, $V(x)$ gets smaller and as a result, air is pushed out. Consider a small time interval dt. In that time interval, h changes by $\dot{h}dt = -\omega L dt$. From the conservation of air particles, $0 = dV(x) + v(x)hx/L$ with $dV(x) = -x^2\omega/2$. Hence,

$$v(x) = \frac{xL\omega}{2h}.$$

Evaluated at $x = L$, this yields

$$v(x = L) = \frac{L^2\omega}{2h}.$$

Grading:

- Stating the idea of using conservation of mass (or implicitly using it) to find the velocity **0.4 pts**
- Equation for the conservation of mass **0.4 pts**
- Correct final expression **0.2 pts**

ii) *(2.5 points)* Since the air flow is laminar and there is no diffusion, all of the lost rotational energy from the falling glass plate will be converted into kinetic energy of air particles. As such, we have $K_{\text{rot}} + K_{\text{air}} = \text{const}$. We found from the previous parts that the air is pushed out at ever faster speeds (between the plates, $v(x) \gg L\omega$) from between the two plates. However, outside of the two plates, the flow will diffuse fast in all directions, such that the vast majority of kinetic energy will be concentrated between the two plates.

The kinetic energy of the air between the two plates is

$$K_{\text{in}} = \int_0^L dx\,\frac{hx}{L}\rho_a\frac{1}{2}v(x)^2$$

$$= \frac{\rho_a L\omega^2}{8h}\int_0^L x^3 dx$$

$$= \frac{\rho_a L^5\omega^2}{32h},$$

and the rotational energy of the glass plate is

$$K_{\text{rot}} = \frac{I\omega^2}{2} = \frac{m_{\text{glass}}L^2}{3}\frac{\omega^2}{2} = \frac{L^3 t\rho_g\omega^2}{6}.$$

Energy conservation then reads as

$$\frac{\rho_a L^5\omega^2}{32h} + \frac{L^3 t\rho_g\omega^2}{6} = \text{const}$$

$$= \frac{\rho_a L^5\omega_0^2}{32h_0} + \frac{L^3 t\rho_g\omega_0^2}{6},$$

and hence,

$$\omega = \omega_0\sqrt{\frac{1 + \frac{3\rho_a L^2}{16\rho_g t}\frac{1}{h_0}}{1 + \frac{3\rho_a L^2}{16\rho_g t}\frac{1}{h}}}.$$

We can see that $\lim_{h\to 0}\omega = 0$, i.e. the air acts as a cushion and stops the glass plate before it hits the stationary plate.

Grading:

- Noticing from the laminarity of the flow that the kinetic energy of the glass plate and its surrounding air is conserved. If the flow is assumed to be dissipative, and the problem is otherwise solved correctly, the maximum score is reduced by **0.3** pts from this part. **0.6 pts**.
- Calculating the kinetic energy of the air, of which: **1.2 pts**,
  - Noting that the majority of the air's kinetic energy is between the two glass plates (no formal proof required) **0.6 pts**.
  - Finding the air velocity at a distance x from the pivot **0.2 pts**.
  - Correct setup for the integral for the kinetic energy **0.3 pts**.
  - Correct final expression for the kinetic energy of air **0.1 pts**.
- Calculating the kinetic energy of the glass slab, of which: **0.4 pts**,
  - Correct expression for the moment of inertia **0.2 pts**.
- Final expression for angular speed as a function of h **0.3 pts**.

iii) *(3 points)* The incoming water from '2' will spread out axisymmetrically along the space between the stone disk and the ceiling. After that, it will spread into the basin and eventually leave through the outgoing pipe '3'. The stone disk is kept up by the pressure differences between the top and bottom of the disc, arising from the flow speed of the water differing on either sides. The fact that pressures differ on boths sides can be seen directly by the application of the Bernoulli Prin-

ciple $p + \rho_w v^2/2 = \text{const}$ along a streamline or by noting that the flow speed gradients are driven by pressure gradients.

The flow speed outside the gap is negligible due to $t \ll R$, hence we can take the pressure at the bottom side to be uniformly p_0. The flow speed in the gap at a distance x from the axis of symmetry can be found from the conservation of mass applied on a concentric cylinder of radius x and height t, giving $2\pi x t \rho_w v(x) = \mu$. Hence,

$$v(x) = \frac{\mu}{2\pi x t \rho_w}.$$

Applying Bernoulli's principle, we get $p(x) + \rho_w v^2/2 = p_0$ and so

$$\Delta p = p_0 - p(x) = \frac{\rho_w v^2}{2} = \frac{1}{2\rho_w}\left(\frac{\mu}{2\pi x t}\right)^2.$$

We can hence find the force due to the pressure differences in the gap by integrating from $x = r$ to $x = R$. First note that the resulting force will be pointing vertically up, because $p(x) < p_0$. Integrating,

$$F_1 = \int_{x=r}^{x=R} 2\pi x \, \mathrm{d}x \Delta p$$
$$= \frac{\mu^2}{4\pi t^2 \rho_w}\int_{x=r}^{x=R}\frac{\mathrm{d}x}{x}$$
$$= \frac{\mu^2}{4\pi t^2 \rho_w}\ln\left(\frac{R}{r}\right).$$

Note that the water entering through the pipe will slow down, pushing the disk further down. The net force from this, however, turns out to be negligible due to the condition $r \gg t$. To see this, one can argue that the said force is of order $\mu v_{\text{pipe}} \sim \mu\mu/(\rho_w r^2) \sim \mu^2/(\rho_w r^2) \ll F_1$.

Further, we have the gravitational force $F_g = -mg = -\pi R^2 h \rho_s g$ pulling the disk down and buoyancy force $F_b = \pi R^2 h \rho_w g$ pushing disk up. The force balance then reads $F_g + F_b + F_1 = 0$. Solving the equation, we find

$$\mu = 2\pi R t \sqrt{\frac{h\rho_w(\rho_s - \rho_w)}{\ln(R/r)}g}.$$

Grading:

- Speed dependence $v(x)$ from continuity condition **0.5 pts**.
- Dynamic pressure from Bernoulli equation **0.5 pts**.
- Express force by integrating dynamic pressure over disk area. **0.5 pts**.
- Explanation why the jet's impact force can be ignored (or equivalently, an expression for the impact force that is carried along in the solution) **0.5 pts**.
- Disk's weight and buoyancy force **0.5 pts**.
- Express answer from force balance **0.5 pts**.

iv) *(0.5 points)* In the context of thermodynamics, entropy is defined in terms of its differential, such that the change in entropy of a system is given by $\mathrm{d}S = \mathrm{d}Q/T$, where $\mathrm{d}Q$ is the heat entering the system, and T its temperature. Further, entropy in reversible thermodynamic processes is a state function, i.e. it only depends on the current (equilibrium) thermodynamical state of the system. This means that when calculating the entropy difference of one mole of vapour and liquid, the temperature at which the phase transition took place does not affect the final result.

As such, it's most convenient to consider the two final states as only differing by the liquid undergoing condensation at $t_0 = 100\,^\circ$C. The final temperature is $t_0 = 100\,^\circ$C because that's when water vapour pressure is equal to p_0 (i.e. boiling temperature at atmospheric pressure). This corresponds to a heat of $\Delta Q = mL = 1\,\text{mol}\cdot ML$ entering the vapour system, compared to the liquid one. Hence, the entropy difference between one mole of vapour and liquid is given by

$$\Delta S = \frac{\Delta Q}{T_0} = \frac{1\,\text{mol}\cdot ML}{T_0} = 110\,\text{J/K}.$$

Grading:

- $\Delta S = \frac{\Delta Q}{T}$ **0.1 pts**.
- $\Delta Q = LM$ **0.1 pts**.
- Understanding that $T = T_0$ **0.2 pts**.

- Correct numerical answer **0.1 pts**.

v) *(3 points)* Because the expansion of water is reversible, entropy is conserved. This means that the change in entropy due to the expansion of the vapour is balanced by the entropy change due to condensation. As discussed before, because entropy is a function of state, it's most convenient to calculate the entropy change by imagining n moles of water (n will later cancel out) cooling and expanding from T_t, p_t to T_0, p_0 and condensing rn moles of water at the end. r is found by demanding that $\Delta S = 0$ in this process.

The entropy change of the vapour is found by applying the first law of thermodynamics over a small temperature and pressure increment $\mathrm{d}T$, $\mathrm{d}p$:

$$\mathrm{d}S_{\text{vapour}} = \frac{\mathrm{d}Q}{T} = \frac{\mathrm{d}U + \mathrm{d}W}{T},$$

where $\mathrm{d}U = nc_v\mathrm{d}T$ is the change in internal energy of the vapour and $\mathrm{d}W = p\mathrm{d}V$ is the work done by the vapour. Importantly, we neglect the volume of water compared to the vapour, as that allows using the ideal gas to simplify the work differential. Using the ideal gas law, we then have

$$p\mathrm{d}V = p\mathrm{d}\left(\frac{nRT}{p}\right) = nR\mathrm{d}T - nRT\frac{\mathrm{d}p}{p}.$$

Hence,

$$\mathrm{d}S_{\text{vapour}} = n(c_v + R)\frac{\mathrm{d}T}{T} - nR\frac{\mathrm{d}p}{p}$$

and we can integrate to get

$$\Delta S_{\text{vapour}} = nc_p\ln\left(\frac{T_0}{T_1}\right) - nR\ln\left(\frac{p_0}{p_1}\right)$$

where $c_p = c_v + R = R(i+2)/2 = R\gamma/(\gamma-1)$ is the heat capacity at constant pressure. The net change in entropy is then

$$\Delta S_{\text{tot}} = 0 = \Delta S_{\text{vapour}} + \Delta S_{\text{condens}}$$
$$= \Delta S_{\text{vapour}} - rn\frac{ML}{T_0}$$
$$= nc_p\ln\left(\frac{T_0}{T_1}\right) - nR\ln\left(\frac{p_0}{p_1}\right) - rn\frac{ML}{T_0}.$$

Thus,

$$r = \frac{RT_0}{ML}\left(\ln\left(\frac{p_1}{p_0}\right) - \frac{\gamma}{\gamma-1}\ln\left(\frac{T_1}{T_0}\right)\right) = 0.114.$$

To find the mass flow rate, we start by finding the flow speed v of the outgoing liquid/vapour. Since we know r, we can apply energy conservation (necessary for the reversibility to hold) on a flowing water packet. This is most conveniently done by demanding energy conservation on the system as a whole. Suppose that in some time interval n moles of water vapour with a volume of V_t are created at the boiler. Then at the outlet, in the steady state, due to conservation of particles, $(1-r)n$ moles of water vapour at a volume of V_0, alongside rn moles of liquid water, are removed. The outflowing water has an additional kinetic energy of $n\mu v^2/2$. The energy change of the whole system due to both steps must cancel each-other out due to energy conservation. We can write this as

$$0 = W_{\text{in}} - W_{\text{out}} + U_{\text{in}} - U_{\text{out}} - nMv^2/2 = 0,$$

where $W_{\text{in}} = p_t V_t = nRT_t$ and $W_{\text{out}} = p_0 V_0 = (1-r)nRT_0$ are the works done by the incoming and outgoing packet and similarly, $U_{\text{in}} = c_v n T_t$ and $U_{\text{out}} = c_v n T_0 - rnML + rnRT_0$ are the internal energies of the incoming and outgoing packet. Notice that for the outgoing internal energy, we have an extra term of $rp_0 V_0 = rnRT_0$. This is because when talking about the latent heat of vaporisation, it includes the work done in order to expand the vapour into the volume it's supposed to occupy. Therefore, we should subtract the said work from the latent heat of vaporisation, in order for it to capture the actual change in the internal energy of the water. Combining everything, we get

$$0 = nRT_t - (1-r)nRT_0 + c_v n T_t$$
$$- c_v n T_0 + rnML - rnRT_0 - nMv^2/2$$
$$= (c_v + R)n(T_t - T_0) + rnML - nMv^2/2,$$

and hence,

$$v = \sqrt{2\left(\frac{c_p \Delta T}{M} + rL\right)} = 906\,\text{m/s}.$$

The density of air at the outlet is found from ideal gas law $\rho = p_0 M/(RT_0)$. The mass flow rate of water vapour is then

$$\mu_{\text{vapour}} = A\rho v = \frac{Ap_0 Mv}{RT_0},$$

but we also have liquid water flowing out, such that the total flow rate is given by

$$\mu = \frac{1}{1-r}\mu_{\text{vapour}} = \frac{Ap_0 Mv}{(1-r)RT_0} = 59\,\text{g/s}.$$

Grading:

Finding r   **1.5 pts**

- Idea that entropy is conserved.   **0.2 pts**
- Idea to calculate entropy change of the whole gas from T_t, P_t to T_0, P_0.   **0.2 pts**
- Equation $dS = \frac{dU+dW}{T}$ **0.1 pts**, and $dS = \frac{cdT+pdV}{T}$ **0.1 pts**.
- Using the ideal gas law to calculate dS using two of the variables T, V, P.   **0.1 pts**
- Integrating **0.2 pts** and expressing entropy difference in terms of P and T **0.2 pts**.
- Using or deriving $c_v = 3R$.   **0.1 pts**
- Correct entropy change due to the phase change   **0.2 pts**
- Express the correct result for r **0.1 pts**

Finding μ   **1.5 pts**

- Idea to use energy conservation along the flow.   **0.2 pts**
- Correct expression for work in the energy conservation.   **0.2 pts**
- Correct internal energy change.   **0.5 pts**
- Correct energy change due to kinetic energy.   **0.1 pts**
- Express correct v **0.1 pts**.
- Equation for gas flow rate $\mu = v\rho_g A$. **0.1 pts**
- The ideal gas law with density   **0.1 pts**
- Correction factor $\frac{1}{1-r}$ for the whole mass flow.   **0.2 pts**

3. ROTATING SPACE STATION (13 points) — *Solution by Kaarel Hänni, grading schemes by Adam Warnerbring, Kaur Aare Saar, Maksim Pokrovskiy and ….*

i) *(0.5 points)* Letting ω be the angular velocity, the acceleration experienced by people on the "ground" is $\omega^2 R = \left(\frac{2\pi}{\tau}\right)^2 R$. Setting this equal to g gives

$$\left(\frac{2\pi}{\tau}\right)^2 R = g \implies \tau = 2\pi\sqrt{\frac{R}{g}} \approx 63.437\,\text{s}.$$

Grading:

- Finding the acceleration at the "ground"   **0.2 pts**
- Correct expression for the period   **0.2 pts**
- Correct numerical answer   **0.1 pts**.

ii) *(2 points)* Let us consider the motion of the ball in the non-rotating (inertial) frame of the center of mass of the spaceship. As the travel time is $t = \tau/2$, the throwing point on the ground will rotate by exactly half a circle between the ball being thrown and the ball being caught. In our inertial frame, the trajectory of the ball will just be a straight line between these two diametrically opposed points; this has length $2R$. In this inertial frame, the ball thus travels with a constant velocity $v_{\text{inertial}} = \frac{2R}{\tau/2} = \frac{4R}{\tau}$ (in the radial direction). The initial velocity vector in the rotating frame is the difference of the velocity vector in the inertial frame and the velocity vector of the throwing point in the rotating frame compared to the inertial frame. So the initial velocity vector in the rotating frame has a radial component of $\frac{4R}{\tau}$ and a tangential component of $\frac{2\pi R}{\tau}$. The throwing speed s is the magnitude of this vector, which is

$$s = \sqrt{\left(\frac{4R}{\tau}\right)^2 + \left(\frac{2\pi R}{\tau}\right)^2} = \frac{2R}{\tau}\sqrt{4 + \pi^2} \approx 117\,\text{m/s}.$$

Grading:

- Finding the trajectory in an inertial frame $\frac{8}{15}$ **pts**
- Finding the radial velocity $\frac{8}{15}$ **pts**
- Finding the tangential velocity $\frac{8}{15}$ **pts**
- Correct velocity addition $\frac{4}{15}$ **pts**
- Correct numerical answer $\frac{2}{15}$ **pts**.

iii) *(2 points)* When the balloon comes to a stop, it is in equilibrium in the rotating frame. An object at radius R' of mass m_1 has a fictitious radial (downward) force of $m_1\omega^2 R'$ acting on it in this frame. This force on the mass m is $m\omega^2(R-H+l)$. The upward force on the balloon is the difference between this force for the balloon and the buoyant force in the rotating frame, which is $\frac{4}{3}\pi r^3(M - M')\frac{n}{V}\omega^2(R-H) = \frac{4}{3}\pi r^3(M-M')\frac{P}{R_G T}\omega^2(R-H)$. Putting all this together, we can write down the condition that the radial force is 0 in equilibrium

$$m\omega^2(R-H+l) = \frac{4}{3}\pi r^3(M-M')\frac{P}{R_G T}\omega^2(R-H)$$

$$\implies m = \frac{\frac{4}{3}\pi r^3(M-M')P(R-H)}{R_G T(R-H+l)} \approx 110.8\,\text{kg}.$$

Grading: If l is neglected, i.e. the mass and balloon are considered to be at equal heights, this problem is marked out of **1.0 pts** maximum. Idea: balance of fictitious radial force and buoyant force **0.8 pts**. Correct expression for force on balloon **0.5 pts** and mass **0.2 pts**. Force balance correct **0.2 pts**. Expression for m correct **0.2 pts**. Numerical answer correct **0.1 pts**.

iv) *(1.5 points)* In the rotating frame, letting r be the distance from the axis of the cylinder, there is an effective radial potential of

$$\varphi(r) = \int_0^r (-\omega^2 x)\ dx = -\frac{\omega^2 r^2}{2},$$

where we have chosen the potential zero level to be at $r = 0$. In other words, in this frame, there is a fictitious radial force

of $m_1\omega^2 r = -m_1\frac{d\varphi}{dr}$ acting on an object of mass m_1. The rope takes a shape that minimizes this potential energy. For this part and the next, we will just be figuring out properties of a rope that minimizes this potential energy (and other than that, we can forget about the rotation). Consider cutting off a tiny piece of rope of length ℓ from point C, then pulling the rope tight at C and gluing it back together (doing work ℓT_C), then cutting the rope open at A, letting it slip by length ℓ (doing work $-\ell T_A$), and finally moving the tiny piece from point C to point A (doing work $\ell\lambda(\varphi(R) - \varphi(R-h)) = -\frac{\ell\lambda\omega^2 h(2R-h)}{2}$), filling the gap of length ℓ. The state of the rope is now the same as initially, so the total work done should be 0:

$$\ell T_C - \ell T_A - \frac{\ell\lambda\omega^2 h(2R-h)}{2} = 0,$$

from where

$$T_A - T_C = -\frac{\lambda 2\pi^2 h(2R-h)}{\tau^2}.$$

See *200 More Puzzling Physics Problems*, problem 78 (and its hint and solution) for a longer explanation of this idea.

Grading: Idea: move piece of rope from C to A/B **0.2 pts**. Idea: work done by tension **0.1 pts** and correct expression **0.3 pts**. Idea: change in radial potential in rotating frame OR kinetic energy in lab frame **0.1 pts** and correct expression **0.3 pts**. Total work done is zero **0.2 pts**. Correct answer for tension difference **0.3 pts**.

v) *(1.5 points)* Let's use the equilibrium condition that torque around the center of the cylinder is 0 (in the rotating frame) for the left half of the rope. Note that the fictitious force is radial, so that contributes nothing to the force. So the only contributions are from the tension in the rope on the two sides, so

$$RT_A \cos\alpha = (R-h)T_C \implies \frac{T_A}{T_C} = \frac{R-h}{R\cos\alpha}.$$

Grading: Idea: rope in equilibrium **0.2 pts**. Using zero total torque as condition **0.2 pts**. Eliminating radial force by choice of point **0.4 pts**. Some correct torque expression containing both tensions **0.2 pts**. Correct ratio **0.5 pts**.

vi) *(1.5 points)* Let the x-axis be the diameter AB, with coordinates chosen to be in meters, and with the coordinates of A being $(-R,0)$. (So the coordinates of B are $(R,0)$ and the coordinates of C are $(0, -(R-h))$. The unique parabola $y = ax^2 + bx + c$ that goes through these three points has $b = 0$ from symmetry across the y axis, $c = -(R-h)$ from considering the point C, and then $a = -\frac{R-h}{R^2}$ from considering the point A. The derivative at A is $\frac{dy}{dx}|_{x=-R} = 2a \cdot R = -\frac{2(R-h)}{R}$, from which

$$\cos\alpha = \frac{-dy}{\sqrt{(dy)^2 + (dx)^2}} = \frac{1}{\sqrt{1 + \left(\frac{dx}{dy}\right)^2}}$$

$$= \frac{1}{\sqrt{1 + \frac{R^2}{4(R-h)^2}}} \approx 0.7106.$$

Using the equations from parts (iv) and (v), we now have a system of equations in two unknowns:

$$\begin{cases} T_A - T_C = -\frac{\lambda 2\pi^2 h(2R-h)}{\tau^2} \\ R T_A \cdot 0.7106 = (R-h)T_C \end{cases}.$$

It remains to solve this system of equations. The second equation gives $T_A = T_C \frac{R-h}{R \cdot 0.7106} \approx T_C \cdot 0.7107$. Plugging this into the first equation then gives

$$T_C = \frac{\lambda 2\pi^2 h(2R-h)}{\tau^2(1-0.7107)} \approx 12631 \text{ N}.$$

Grading: Some correct parabola given some choice of axes **0.3 pts**. Correct expression for $\cos\alpha$ **0.2 pts** and correct numerical value **0.2 pts**. System of equations using expressions found in iv) and v), even if they are incorrect, **0.2 pts**. Correct expression

for tension **0.4 pts**. Solved only for T_A: full marks with penalty **-0.1 pts**. Correct numerical value for T_C **0.2 pts**.

vii) *(2 points)* The rotating charge on the walls is making the spaceship into a solenoid carrying current $I = \frac{Q}{\tau}$. Inside of the spaceship, this creates a constant axial magnetic field of magnitude

$$B = \mu_0 \frac{I}{L} = \mu_0 \frac{Q}{\tau L}.$$

The force this creates on the charged ball is $-qvB = -q\frac{2\pi R}{\tau}\mu_0\frac{Q}{\tau L}$ in the radial upward direction. For the ball to hover above the "ground" motionlessly, the acceleration created by this radial force has to be equal to the centripetal acceleration $\omega^2 R = \frac{(2\pi)^2 R}{\tau^2}$:

$$-\frac{q}{m}\frac{2\pi R}{\tau}\mu_0\frac{Q}{\tau L} = \frac{(2\pi)^2 R}{\tau^2} \implies \frac{q}{m} = -\frac{2\pi L}{\mu_0 Q}.$$

Grading:
Method 1:

· Finding the current $\frac{4}{15}$ **pts**
· Finding the magnetic field   **0.4 pts**
· Finding the upward force on the charged ball $\frac{2}{3}$ **pts**
· Writing out the centripetal acceleration of the ball $\frac{4}{30}$ **pts**
· Correct relation between force and acceleration $\frac{4}{15}$ **pts**
· Correct final expression $\frac{4}{15}$ **pts**
· Wrong sign of the charge $-\frac{4}{15}$ **pts**

Method 2:

· Finding the current $\frac{4}{15}$ **pts**
· Finding the magnetic field   **0.4 pts**
· Finding the upward force on the charged ball $\frac{2}{3}$ **pts**
· Writing out the force balance in the rotating frame (with the correct expression of the upward force)   **0.4 pts**
· Correct final expression $\frac{4}{15}$ **pts**
· Wrong sign of the charge $-\frac{4}{15}$ **pts**

viii) *(2 points)* Consider a charge q_1 at rest at radius r in the rotating frame. The electromagnetic force applied to it is frame-independently $q_1\frac{2\pi r}{\tau}\mu_0\frac{Q}{\tau L}$. This charge is not moving in the rotating frame, so in its frame, the force applied to it by the magnetic field must be 0, and so the electric field measured in its frame must satisfy $q_1 E = q_1\frac{2\pi r}{\tau}\mu_0\frac{Q}{\tau L}$, from where

$$\vec{E} = \frac{2\pi\mu_0 Q}{\tau^2 L}\vec{r}.$$

We now have an expression for E at each point in the rotating frame; what remains is to evaluate $\oint \vec{E} \cdot d\vec{A}$. For the sake of variety, we will demonstrate two ways to evaluate this integral.

For the first option, note that our $\vec{E} = const \cdot \vec{r}$ is exactly the electric field of a uniformly charged cylinder (with the correctly chosen charge density), and so we could equivalently find the same integral around such a uniformly charged cylinder, which Gauss' theorem gives as $V\rho/\epsilon_0 = V \cdot c_1$. We can find an expression for the constant c_1 by considering the simple case where the surface we are dealing with is a cylinder of width 1 m itself, in which case the integral is

$$\oint \vec{E} \cdot d\vec{A} = 1 \text{ m} \cdot 2\pi r \frac{2\pi\mu_0 Q}{\tau^2 L}r = V\frac{4\pi\mu_0 Q}{\tau^2 L}.$$

Hence, $c_1 = \frac{4\pi\mu_0 Q}{\tau^2 L}$, and so

$$\oint \vec{E} \cdot d\vec{A} = V\frac{4\pi\mu_0 Q}{\tau^2 L}.$$

To briefly describe a second option for evaluating this integral, note that by partitioning a 3D body into volume slices, with each slice bounded by an area element $d\vec{A}$ and with cylindrical radius vector $\vec{r}$, and with the volume of each slice being $\vec{r} \cdot d\vec{A}/2$, we get that the volume of the body is the sum of volumes of all such slices,

$$V = \oint \frac{1}{2}\vec{r} \cdot d\vec{A}.$$

This lets us evaluate the main integral as

$$\oint \vec{E} \cdot d\vec{A} = \oint \frac{2\pi\mu_0 Q}{\tau^2 L}\vec{r} \cdot d\vec{A}$$

$$= \frac{4\pi\mu_0 Q}{\tau^2 L} \oint \frac{1}{2}\vec{r} \cdot d\vec{A} = \frac{4\pi\mu_0 Q}{\tau^2 L}V.$$

Grading:

· Arguing that in the rotating frame there is only an electric field   **0.4 pts**
· Finding the electric field strength   **0.4 pts**
· Choosing a suitable Gauss surface   **0.4 pts**
· Evaluating the surface integral and obtaining the correct answer   **0.8 pts**.

4. STRETCHING GLOVES (8 points) — *Solution by Eero Uustalu, grading schemes by ….*

i) *(1 point)*

Cut a few strips of the material with the same constant width and mark them with two perpendicular lines close to the ends of the strip while leaving ample space for affixing or binding. Then roll one strip lengthwise into a cylindrical shape for stretching. Measure the initial length between the marked lines l_0. With a straight measuring tape affixed to the surface of the table, stretch one of the strips to the breaking point while determining the maximum length between the lines before the strip breaks l_{max}. Repeat this at least three times to see if the data is reproducible. Calculate the result $\epsilon_{max} = (l_{max} - l_0)/(l_0)$.

In our measurement we got $\epsilon_{max} = 5.5$.

Grading:

· Selection of the actual latex piece for the sample. Strips have to be of high quality and not include the finger end.   **0.1 pts**
· Selecting the strip length: **0.2 pts**
   – at least 8cm: **0.2/0.2 pts**
   – at least 4cm: **0.1/0.2 pts**
   – shorter, not specified   **0/0.2 pts**
   – *Remark*. If a series of different lengths of strips is used, then the medium value should be considered. Additionally, if

the graph for finding ϵ_{max} is used by including the point $(0,0)$, then **0.1 pts** should be subtracted.

- Method of measurement, of which: **0.4 pts**
  - one end affixed with tape or held using a ruler **0.1 pts**.
  - lines marked on the strip, or the strip is affixed over the edge of the ruler **0.1 pts** (if both ends are affixed, **0.2 pts**, but can't get more than **0.4 pts** for this section).
  - using narrow strips **0.1 pts**.
  - measuring only the part of the strip that's being stretched (between marks) **0.1 pts**.
  - Measuring the full length of the stip **0.1 pts**.
- Repetition of measurements: **0.2 pts**
  - measurement repeated at least 5 times **0.2/0.2 pts**.
  - measurement repeated at least 3 times **0.1/0.2 pts**.
  - if distortion of measured ϵ_{max} values is low **+0.1 pts**.
- Final numerical value. If the value lies in range 4-7 (depends on the material, even 3-11 may be ok) **0.1 pts**.

ii) *(7 points)*

Theoretical considerations.

We can assume the material has uniform thickness. Knowing that $\sigma = \frac{F}{A}$, $V = \text{const}$, and that any force applied to the strip affects all directions perpendicular to the applied force equally, then at any time for the same strip

$$V = a \cdot a \cdot k \cdot l = a_0 \cdot a_0 \cdot k \cdot l_0 = a_{max} \cdot a_{max} \cdot k \cdot l_{max},$$

where a_0 is the initial thickness of material with no force applied and l_0 is the initial length of the strip with no force applied, a is the thickness when force F is applied and l is the length of the strip when force F is applied, a_{max} and l_{max} are the respective values at the breaking force F_{max}.

k is the ratio between the width d and

the thickness a of the strip such that $d = k \cdot a$ where k remains constant for the strip while stretching. Therefore

$$\sigma = \frac{F}{a \cdot a \cdot k} \quad \text{and} \quad \sigma_{max} = \frac{F_{max}}{a_{max} \cdot a_{max} \cdot k}.$$

From $V = \text{const}$ we can deduce that $a \cdot a = \frac{a_0 \cdot a_0 \cdot l_0}{l}$ and $a_{max} \cdot a_{max} = \frac{a_0 \cdot a_0 \cdot l_0}{l_{max}}$.

If we could apply the same force F on strips of different initial widths $d = k \cdot a$, we could vary the tension for each strip. Therefore we would be using different strips with different k values for each measurement

$$\sigma = \frac{F}{k} \cdot \frac{l}{l_0} \cdot \frac{1}{a_0 \cdot a_0} \quad \text{and} \quad \sigma_{max} = \frac{F}{k_{max}} \cdot \frac{l_{max}}{l_0} \cdot \frac{1}{a_0 \cdot a_0},$$

where $k = d/a$ and $k_{max} = d_{max}/a_{max}$ are the width to thickness ratios of the the respective strips used for F and F_{max}. By combining the above, we get

$$\frac{\sigma}{\sigma_{max}} = \frac{k_{max}}{k} \cdot \frac{l}{l_{max}},$$

but since $d_0 = k \cdot a_0$ is the initial unstressed width of one strip and $d_{max0} = k_{max} \cdot a_0$ is the initial unstressed width of another strip stretched to the breaking point, for building the graph it transforms to:
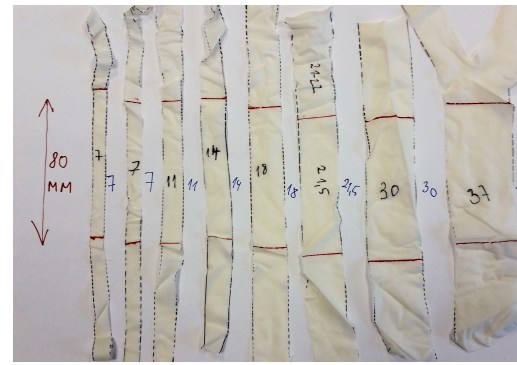
$\frac{\sigma}{\sigma_{max}} = \frac{d_{max0}}{d_0} \cdot \frac{l}{l_{max}}$ for the Y axis and

$\epsilon = \frac{l - l_0}{l_0}$ for the X axis.

Measurements.

We can assume the material has uniform thickness. We want to apply the same force on the strips with different initial widths.

Approach 1

We make many strips with different widths, mark them with the same initial length l_0 using perpendicular lines, measure and record the initial widths d_0 of each segment, then roll them lengthwise into a cylindrical shape and bind them one after another. The force will be the same for all the segments, but the value of σ for each segment will depend on the initial width of each individual strip.

A range where d_{min} and d_{max} differ at least 8 times is recommended.

Rolling the strip lengthwise into a cylindrical shape before binding gives the possibility to distribute the force evenly on all the strips regardless of the width of the strip!

Our solution was to make two identical strips with the smallest possible width and attach them to one another at one end to form a Y shape. The previously determined ϵ_{max} was used to estimate the possible maximum stretching almost to the breaking point of the narrowest strip. In case of a breakage event the identical spare strip at the other end of the Y shape could be used for measurement. After affixing the multi-width rope of combined strips at its maximum stretch position, the values of l were recorded for each individual section. Afterwards the rope was stretched till breaking to find the actual breaking length l_{max} of the narrowest strip.

The **7 mm** wide initially **80 mm** long test strip broke at length **522 mm**. Therefore the

measurement of the rope of strips was made using force F stretching the second **7 mm** wide **80 mm** test strip to **519 mm**.

But the **7 mm** stripe broke when it was stretched to **599 mm**.

So the $d_{max0} = 7\,\text{mm}$ and $l_{max} = 599\,\text{mm}$.

$(l - l_0)/l_o$	l_0 mm	l mm	d_0 mm	σ/σ_{max}
0.638	80	131	71	0.022
1.606	80	208,5	37	0.066
1.888	80	231	30	0.090
3.025	80	322	21,5	0.175
3.338	80	347	18	0.226
4.225	80	418	14	0.349
4.925	80	474	11	0.505
5.488	80	519	7	0.866

And the graph, $\frac{\sigma}{\sigma_{max}}$ for the Y axis and ϵ for the X axis.
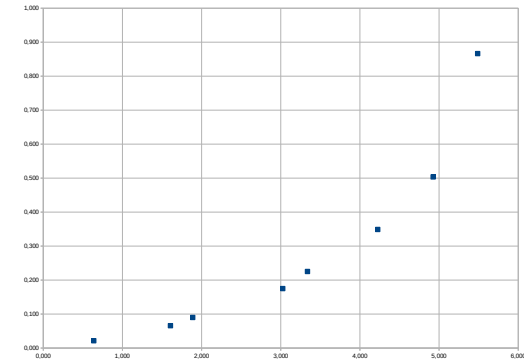
Approach 2

We make two identical elongated triangles from the longest piece of material available. We mark the triangles with evenly spaced perpendicular lines. (The lines are perpendicular to the central symmetry line of the elongated triangle.)
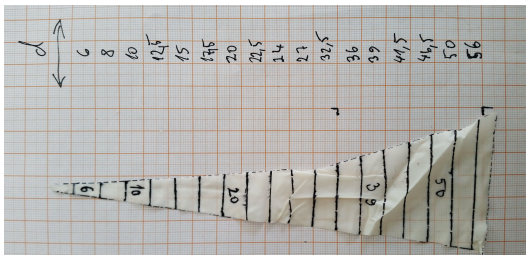
We record the initial length l_0 and the average width d_0 of each segment as defined by the perpendicular lines. When stretched, all segments will have the same force applied but each segment will have its own σ. We have to affix the base of the prolonged triangle to the ruler with tape (it has to be a good strong joint) and we affix the ruler to the table. A long strip of graph paper was taped to the table to have the rubber triangle stretched over it.
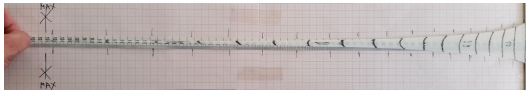

The triangle was stretched until the narrow end segment was near breaking point, and the values of l were recorded for each segment. (Most easily done by using a marker on the graph paper while holding the narrow end of the triangle at a fixed position with the other hand). One of the triangles was actually stretched to the braking point, the other triangle was stretched to the breaking point only after the measurement was made.

The narrow end segment of the first triangle of length $10\,\text{mm}$ and of average width $6\,\text{mm}$ broke at length $73.1\,\text{mm}$. Therefore the measurement of the second similar triangle tested was made using force F stretching the narrow end segment of the triangle to $72.6\,\text{mm}$. The actual breaking of the narrow segment of the second triangle occurred at length $73.7\,\text{mm}$.

So the $d_{\max 0} = 6\,\text{mm}$ and $l_{\max} = 73.7\,\text{mm}$.

$\sigma/\sigma_{\max}$	mm l_0	mm l	mm d_0	$(l-l_0)/l_o$
0.985	6	10	72.6	6.26
0.605	8	10	59.5	4.95
0.444	10	10	54.5	4.45
0.333	12.5	10	51	4.1
0.266	15	10	49	3.9
0.221	17.5	10	47.5	3.75
0.183	20	10	45	3.5
0.148	22.5	10	41	3.1
0.129	24	10	38	2.8
0.104	27	10	34.5	2.45
0.074	32.5	10	29.5	1.95
0.061	36	10	27	1.7
0.051	39	10	24.5	1.45
0.043	41.5	10	22	1.2
0.037	46.5	10	21	1.1
0.030	50	10	18.5	0.85
0.022	56	10	15	0.5

And the graph, $\frac{\sigma}{\sigma_{\max}}$ for the Y axis and ϵ for the X axis.
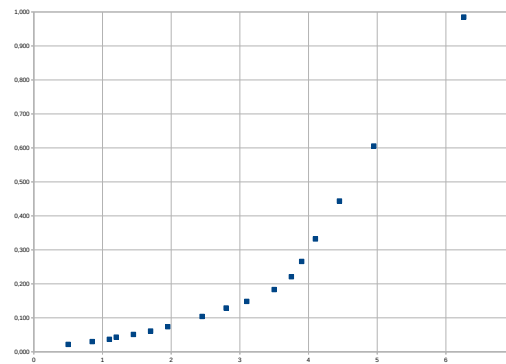
Grading:

GOOD practice:

- using the longest possible strips
- marking the initial length on the strip with a marker or
- measuring initial length to one straight affixed surface to another parallel affixed surface
- rolling the marker-marked strip lengthwise into a cylindrical shape for uniform stretching
- reasonable amount of repetitions (breaking is not very well repeatable)

  BAD practice:

- intentionally variable length strips / short strips
- not marking the initial length on the strip with a marker
- not stretched with perpendicularly uniform strain
- not taking measurements only from actual part to be stretched
- not accounting for possible friction (some methods)

  part ii

  other solutions proposed:

- multiple similar masses used (not in the equipment list)
- a lever and some mass
- from the same latex material (a very long uniform and relatively thick strip is needed to be in proportional range, but then the measured force will be low)
- multiple similar strips in parallel as opposed to using fever strips in row and breaking with even force
- trigonometry and constant force
- different width strip pairs (force applied must remain constant for all pairs, in some solutions friction would need to be accounted for)

  unsuccessful solving attempts / bad practices:

- using equipment not listed
- calculated, not measured (experiment!) **0 pts**.
- using a finger as a force measuring device **0 pts**.
- assuming the repeatedly applied force is constant (which it might not be) **0 pts**.
- measuring by varying the length of the strip while keeping the applied force constant (stress created remains constant) (force may be mg) **0 pts**.
- stretching the strip to different lengths not knowing the actual force applied **0 pts**.
- MOST OFTEN: Assumption that F=kx ! That is actually an alternative lightly modified form of writing down the dependency WE HAVE TO MEASURE, not assume! (and the real dependency is not linear) **0 pts**.

  Possible grading scheme for the solutions given as example:

  part ii

  theory:

- assuming that the material has initial uniform thickness **0.2 pts**.
- idea that as $V = const$ then the dimensions in axes perpendicular to the applied force change in a fixed ratio to each other **1.0 pts**.
- idea of changing σ by changing the width d of the material when constant force is applied **1.0 pts**.
- idea of applying a common force to all the different samples at the same time **1.0 pts**.
- finding how to calculate the value of ϵ to be plotted **0.2 pts**.
- finding how to calculate the value of $\sigma/\sigma_{\max}$ to be plotted **0.6 pts**.

  measurements and data:

- the strips/intervals used are as long as possible (in our case at least 14cm) **0.1 pts**.
- data range covered - value of the largest measured σ differs from the smallest measured σ at least 8 times **0.1 pts**.
- data points - at least 8 points measured (**0.2 pts** for each correctly measured point or **0.1 pts** for each correctly measured point if less than 4 points measured (have you seen graph with 3 data points?), for maximum 8 measured points) MAX **1.6 pts**.
- data evenly distributed **0.1 pts**.
- at least one data point measured with a σ value at least 0,8 of $\sigma_{\max}$ **0.1 pts**.
- measurements for the breaking event values correctly done **0.2 pts**.

- chart of measured and calculated values given **0.3 pts**.

graph:

- graph correctly plotted with all measured points **0.3 pts**.
- graph plotted in units asked (ratios, no units) **0.2 pts**.

If the results of any intermediate calculations are excessively rounded, a deduction from the final score of **−0.1 pts** for every intermediate stage used.

using out of list equipment:

- If the out of list equipment used does not allow for any principally new methods of solution, any credit awarded for the parts of the solution that rely on the use of the out of list equipment or data measured with it will be reduced by 50%.

- If the out of list equipment does enable a principally new method of solution, credit will only be awarded for aspects of the solution that would be applicable in a solution that does not rely on the out of list equipment.