NORDIC-BALTIC PHYSICS OLYMPIAD 2019

1. SATELLITE (8 points) — *Taavet Kalda*. A solar powered satellite with starting velocity v_0 is launched from Earth onto an elliptical heliocentric orbit with the intent of gathering as much solar energy as possible. The angle of departure can be freely changed.

i) *(1 point)* What is the minimal required launching speed v_m for the satellite to reach any heliocentric orbit?

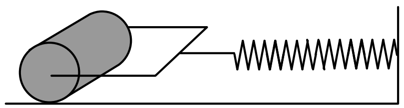
ii) *(2 points)* What is the speed of the satellite just after exiting Earth's gravitational field?

iii) *(2.5 points)* Express the long-time-average solar irradiance of the satellite in terms of its semi-major axis a, angular momentum J, orbital period T and mass m.

iv) *(2.5 points)* What is the maximal average solar irradiance the satellite can gather and what is the needed launch angle relative to Earth's motion?

The Sun's mass is $M_\odot$, Earth's orbital radius $R_\oplus$, free fall acceleration on Earth g, Earth's radius $r_\oplus$ and Sun's luminosity $L_\odot$.

2. ROLLER (8 points) — *Lasse Frantti (iv,v: Jaan Kalda)*. A roller consists of a solid homogeneous cylinder of mass M and radius r; it rests on a horizontal table and is attached to a wall via a helical spring of spring constant k (see figure). The spring can be assumed to be of a negligible mass and ideal, i.e. the Hooke's law remains valid for arbitrarily large deformations.



i) *(1 point)* At first, let as assume that there is no friction between the cylinder and the table. The roller is pushed aside and released; find the period of oscillations T_0.

ii) *(1 point)* From now on, the coefficient of friction between the roller and the table μ is no longer neglected. The roller is pushed aside and starts to roll back and forth. For small oscillation amplitudes, there is no slipping between the roller and the surface. Find the new period of oscillations T_r.

iii) *(2 points)* If the initial oscillation amplitude (measured as the deformation x of the spring) is larger than a certain critical value $A_\star$, the amplitude of oscillations starts decreasing in time. Express $A_\star$ in terms of k, M, r, the free fall acceleration g, and.

iv) *(2 points)* Assuming that the initial amplitude A_0 is much larger than $A_\star$, what is the maximal angular speed of the cylinder during $0 \le t \le T/2$, where t is the time elapsed since releasing the roller?

v) *(2 points)* Assuming that $A_0 \gg A_\star$, sketch a qualitative graph showing the dependance of εr and a as functions of time; here ε and a denote the angular acceleration and linear acceleration of the roller, respectively.

3. MOTION IN B (8 points) — *Andréas Sundström, Joonas Kalda (ii,iii)*. Particles with mass m and charge q are launched from the origin with speed v, parallel to the x-axis. There is a screen at $x = l$.

i) *(1 point)* The first particle is launched when there is an homogeneous electric field parallel to the x-axis, and there is no magnetic field. What should be the strength of the electric field so that the particle would never reach the screen?

ii) *(2 points)* Next, the electric field is turned off, a homogeneous z-directional magnetic field in region $l > x > 0$ is swithced on, and the second particle is launched. Knowing that particle's speed is just big enough to reach the screen, sketch the trajectory and find the magnetic fied strength B.

iii) *(2 points)* Finally, an electric field lying in the x,y-plane is turned on while B is kept unchanged. Third particle is fired from the origin, still parallel to the x-axis, but possibly with a different speed. It is observed that the particle travels without deviation. Further, the time it takes for it to reach the screen is the same as for the second particle. Find the magnitude of the electric field E.

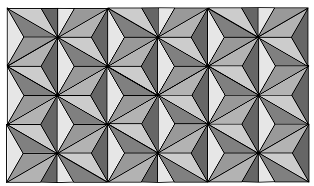
iv) *(3 points)* Unrelated to the previous tasks, let us study now a weakly inhomogeneous magnetic field: the magnetic field lines have a radius of curvature which is much larger than the radius of curvature of the particle's trajectory. It appears that in that case, the so-called adiabatic invariant of a particle in magnetic field is conserved: the magnetic flux embraced by the particle's helix-shaped trajectory remains constant with an extremely good precision along the trajectory.

Consider a very simplified model for how a solar wind particles interact with the Earth's magnetic field. The strength of the Earth's magnetic field on its magnetic axis can be expressed as $B(z) = B_E(R_E/z)^3$, where $B_E = 3.12 \times 10^{-5}\,\mathrm{T}$ is the field strength at the Earth's surface on the magnetic pole, $R_E = 6370\,\mathrm{km}$ is the radius of the Earth and z is measured from the *center* of the Earth.

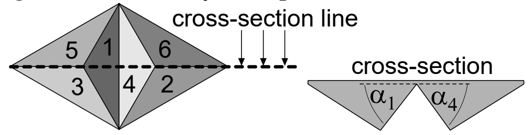
An electron of charge $-e = -1.60 \times 10^{-19}\,\mathrm{C}$ and mass $m_e = 9.11 \times 10^{-31}\,\mathrm{kg}$ is approaching the Earth with speed $u_0 = 500\,\mathrm{km/s}$ and hits the Earth's magnetic field right on its axis at a distance $R_0 = 5R_E$ with an angle α and starts spiraling in towards the Earth.

If α is too large, the particle will be reflected by the increasing magnetic field strength as the particle approaches Earth. Find the condition on α for the particle to reach the surface of the Earth. You can neglect any gravitational and/or relativistic effects.

4. RETROREFLECTIVE FILM (12 points) — *Eero Uustalu and Jaan Kalda*. You are given the following tools: a retroreflective film an enlarged bottom view of which is shown in the first figure; stand, ruler, laser pointer, screen, graph paper, protractor.



While the top surface of the film is flat, the bottom surface is a periodic array of slanted triangular faces. Six such faces are shown enlarged in the seond figure; the faces 1, 3 and 5 are perpendicular to each other and form a corner of a cube, and the faces 2, 4 and 6 are also perpendicular to each other. On the right of the second figure, a cross-section of the film is shown. The film's material between the slanted faces and the flat surface form microprisms. The refracting angles of these microprisms are denoted by α_i, $i = 1,2,\ldots 6$ (the index numbers correspond to those of the faces). Among the angles α_i, some may be equal to each other.



When light falls onto the flat surface close to perpendicular incidence, it undergoes total internal reflections on the slanted faces, and as a result, its direction of propagation is rotated by $180°$. However, the microprisms can also serve as prisms diverting a light beam by an angle β. The angle β depends on the angle of incidence, and on the prism angle $\alpha = \alpha_i$. Let β_i denote the minimal deflection angle for a fixed prism angle α_i.

i) *(2 points)* Perform experiment to relate the prism angles α_i (with $i = 1,2,\ldots 6$ relating to the face number as shown in the figure) with equalities and inequalities. You may use the deduced equalities throughout this problem (reducing thereby the number of unknown angles). Note that the face corresponding to the index "1" can be chosen arbitrarily.

ii) *(2 points)* Determine the minimal deflection angles β_i, $i = 1,2,\ldots 6$.

iii) *(4 points)* Determine the prism angles α_i, $i = 1,2,\ldots 6$.

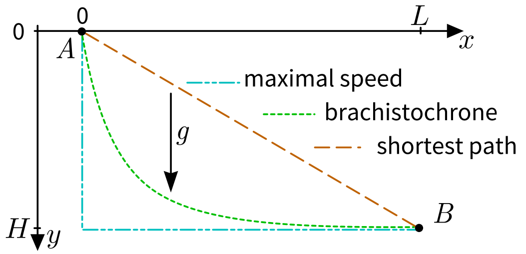
iv) *(1 point)* Due to the fact that the faces 1, 3 and 5 are perpendicular to each other, the following equality holds:

$$\cos^2\alpha_1 + \cos^2\alpha_3 + \cos^2\alpha_5 = 1.$$

Similarly, $\cos^2\alpha_2 + \cos^2\alpha_4 + \cos^2\alpha_6 = 1$. Use these equalities to adjust the result for the prism angles by adding and/or subtracting the same small value from your previously obtained angles' values.

v) *(3 points)* Determine the refraction coefficient of the film's material.

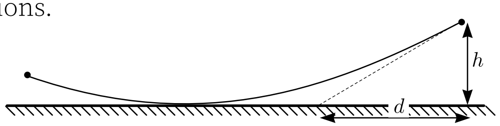
5. Brachistochrone (10 points) — *Rūdolf Treilis*. Consider points A and B separated by height H in the vertical direction and distance L in the horizontal direction, placed in a gravitational field g as shown in the figure below. A point mass can slide along a rail of fixed shape frictionlessly (including taking $90°$-turns) from A to B. The brachistochrone curve is the curve minimizing the total travel time.



i) *(2 points)* Calculate the total travel time for the "maximal speed" and "shortest path" trajectories. Find the ratio $\frac{L}{H}$ for which the two are equal.

ii) *(2 points)* According to Fermat's principle, light ray travels from one point to another along the path of shortest travel time. Suppose that in a certain medium, a light ray can propagate from A to B along the brachistochrone curve shown in the figure above. Find the refractive index $n = n(x,y)$ as a function of the coordinates x and y for this medium if $n(L,H) = 1$.

iii) *(2 points)* Show that the path of a light ray traveling in a medium with a variable refractive index $n(x,y) \equiv n(y)$ satisfies the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{C \cdot n(y)^2 - 1}$, where C is a constant determined by boundary conditions.



iv) *(2 points)* The obtained equation can explain mirages, which occur when the index of refraction increases with height. Consider a light ray coming from the sky that grazes the surface of the Earth ($y = 0$) and hits the eye of an observer at height h (for this task choose the y-axis in the opposite direction - bottom of the page to top). If the refractive index varies as $n(y) = n_0(1 + \alpha y)$ with n_0 and α constant, find an expression for the apparent distance that the ray of light is emanating from d.

v) *(2 points)* Solving the equations derived in parts ii) and iii), one may show that the brachistochrone curve is actually a segment of a cycloid. A cycloid is the curve traced by a fixed point on the rim of a circular wheel as it rolls along a straight line without slipping. For the special case $\frac{L}{H} = \frac{\pi}{2}$ find the minimum travel time $t_{\min}$ between A and B.

6. Self-gravitating gas (10 points) — *Eero Vaher (v: Jaan Kalda)*. Consider a ball of monoatomic ideal gas at temperature T which keeps a spherically symmetric mechanically stable stationary shape due to its own gravity field. Let the total mass of the gas be M_0, and its molar mass μ. We shall describe the radial mass distribution in terms of the total mass $M = M(r)$ inside a sphere of radius r, and the pressure $p = p(r)$ as a function of distance r from the centre of the spherical gas cloud.

i) *(2 points)* State the mechanical equilibrium condition for a small parcel of gas of mass m and volume v at distance r from the centre of the cloud in terms of the local pressure gradient $p'(r) = \frac{\mathrm{d}p}{\mathrm{d}r}$, the local value of $M(r)$, and the appropriate constants of nature. Simplify your expression by noticing that $m/v = \rho$ is the local density of the gas.

ii) *(2 points)* Show that the total thermal energy of the gas can be expressed as

$$U = -\alpha \int V \,\mathrm{d}p,$$

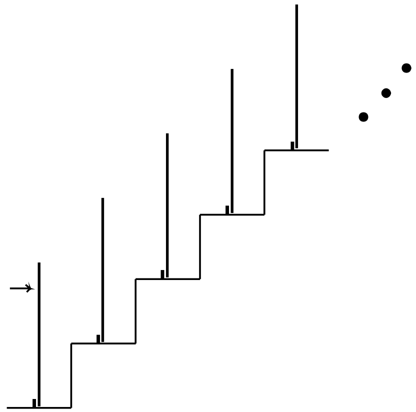
where $V = \frac{4}{3}\pi r^3$, and α is a numerical factor, and the integral is taken from the centre of the cloud, to very far distances from the centre, where the pressure is negligibly small. Find the value of α.

iii) *(3 points)* Based on your previous results, show that $U = -\beta E_G$, where E_G is the gravitational potential enery of the gas, and $\beta < 1$ is a positive numerical factor. Find the value of β.

iv) *(1 point)* How will the temperature and characteristic radius of the cloud change in time due to heat radiation? Give a qualitative motivated answer. (The characteristic radius R_c can be defined as such a radius that half of the mass of the cloud is inside the sphere of radius R_c.)

v) *(2 points)* Consider now a similar fully ionized plasma cloud. Assume that plasma is a macroscopically neutral mixture of electrons and protons. What is the proportionality factor between the total gravitational energy and total thermal energy for the plasma cloud?

7. Dominoes (6 points) — *Kaarel Hänni*.



David stands at the bottom of an infinite staircase with both step width and height being equal to d. The corner of each step is slightly rounded. In the middle of each step, there is initially an upright domino of length $\sqrt{5}d$ and negligible thickness. Behind the base of each domino, there is a small ridge that prevents it from sliding backward. David gives the first domino some initial angular velocity, and the dominoes start falling into each other. All collisions are perfectly inelastic, and there is no friction between two dominoes. David notices that after a while, all dominoes have equal initial angular velocity ω. Find ω.

8. Four resistors (10 points) — *Jaan Kalda and Eero Uustalu*. *Tools:* four almost identical resistors (of resistance slightly more than $4\,\mathrm{k}$) labelled with letters A–D and the set number, multimeter, voltage source with adjustable output voltage, wires. **Do not use the ammeter function of the multimeter**; if you still do use, and burn the multimeter, you will not be given another multimeter. The multimeter has a four-digit display, but the first digit can be only 0, 1, 2 or 3. When the multimeter is used as ohmmeter, the uncertainty is 1.0% of the reading plus 4 times the smallest digit value. When the multimeter is used as voltmeter, the uncertainty is 0.6% of the reading plus 4 times the smallest digit value.

i) *(2 points)* Write down your set number. Determine the average resistance $\bar{r}$ of the four resistors as precisely as possible and state the uncertainty. Draw the circuit you used.

ii) *(2 points)* Determine the harmonic average resistance $\langle r\rangle$ of the four resistors as precisely as possible (harmonic average is the reciprocal of the average of reciprocals) and state the uncertainty. Draw the circuit you used.

iii) *(1 point)* Arrange the resistors A, B, C and D in increasing order of resistances. Draw the circuit you used.

iv) *(5 points)* Determine $r_A - \bar{r}$, $r_B - \bar{r}$, $r_C - \bar{r}$, and $r_D - \bar{r}$ as precisely as possible and state the uncertainty (r_A, r_B, r_C, and r_D denote the resistances of the respective resistors). Draw the circuit you used.