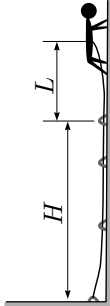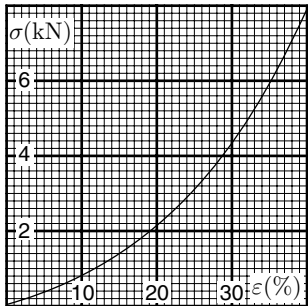## 1. Rock Climber (6 points)

A rock climber of mass $m = 80$ kg ascends along a vertical rock. For self-protection, the climber uses the following method. One end of an elastic rope is anchored to the ground. The rope goes through smooth protection loops (carabiners), which are anchored to the rock. The height of the last carabiner is $H = 20$ m. The other end of the rope goes through a special braking clip which is tied to the harness of the climber. During the climb, this clip keeps rope tight, but enables the climber to lengthen protective part of the rope. (Assume that the rope between the clip and carabiners is always tight) *When falling, the maximum acceleration must not exceed $a_{max} = 5g$* (to protect from injuries). You may assume that the rope is always vertical, the distance between the clip and the centre of mass of the climber is very small, and friction between the rope and carabiners is negligible. Relationship between the strain and stress of the rope is sketched on the graph below.
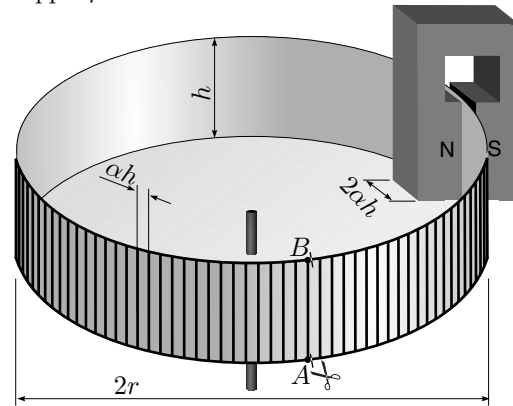
1) Assume that the distance between the climber and the last carabiner is L (see Figure). If the climber happens to fall, the distance between the highest carabiner and the climber will reach a maximal value l (afterwards, the elasticity of the rope starts lifting the climber). Which inequality should be satisfied for l ? (1.5 pts)

2) Find the maximal safe length L between the climber and the last carabiner (upon reaching of which he has to anchor a next carabiner; 4.5 pts).

2. Magnetic break (12 points)

Foucault currents can be used to decelerate moving metal objects, e.g. a rotating disc. Consider the following simple model. For the sake of simplicity, the disc is substituted by a "circular railway", see Figure. Plastic "can" of radius $r = 15$ cm, mass $m = 100$ g, and height $h = 1$ cm consists of a homogeneous disc (the bottom of the "can"), and of a much thinner cylindrical wall. Along the edges of the cylinder, there are two wire rings ("rails"), which are connected with a set of parallel wire bars ("sleepers"). Both the "sleepers" and "rails" are made of a copper wire of diameter $\delta = 0.2$ mm; the distance between the "sleepers" $L = \alpha h$, where $\alpha = 0.3$. Frictionless rotation of the system is decelerated with an homogeneous magnetic field ($B = 1$ T) in the slit between the poles of a permanent magnet, see Figure. Assume that the homogeneous field fills a region of rectangular cross-section, equal to the area between three subsequent "sleepers" (i.e. of size $h \times 2\alpha h$). Outside of that region, the field is negligible. The specific resistance of the copper $\rho = 1.724 \cdot 10^{-8}$ Ωm.

Suppose the system is motionless.

1) Find the resistance R of a single "sleeper" (1 pt).

2) Suppose the "rails" are cut near the end-points of a certain "sleeper" A and B. Prove that the resistance between A and B is approximately given by $R_0 = R[\sqrt{\alpha(\alpha + 2)} - \alpha]$ (2 pts).

Now suppose the system is rotating with an angular speed $\omega = 1$ rad/s, and the "rails" are uncut.

3) Sketch an equivalent DC circuit, so that the currents through the resistors are equal to the currents in the respective "railway" elements: "sleepers" and "rail" segments (between subsequent "sleepers"; 2 pts).

4) Using the above obtained results, prove that the (Joule) dissipation power is given by formula $P = kB^2\omega^2/R$, and express the constant k (3 pts).

5) Find the decelerating torque M (2 pts).

6) Prove that the angular speed will vanish as $\omega = \omega_0 e^{-t/\tau}$, and determine the time constant τ (2 pts).

3. Ballistic rocket (8 p)

A rocket is launched from a pole of the Earth with the first cosmic velocity (near-Earth orbital velocity) in such a way that it lands at the Equator. The radius of the Earth $R = 6400$ km.

1) Find the longer semi-axes a of the rocket's orbit (1.5 pts).

2) What is the maximal height of the rocket's orbit h (from the Earth's surface; 3.5 pts)?
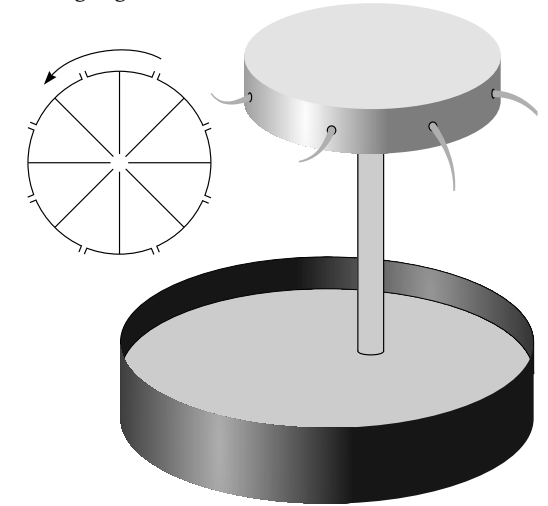
3) What is the rocket's flight time τ (3 p)?

Remark: The mechanical energy of a planet rotating around a star $E = -GMm/2a$, where G is the gravitation constant, M — the star mass, m — the planet mass, and a — the orbit's longer semi-axes (zero potential energy corresponds to an infinite departure). The surface area of an ellipse $S = \pi ab$, where b is the shorter semi-axes.

4. Water pump (10 p)

Consider the following construction of a water pump. A vertical tube of cross-sectional area S_1 leads from an open water reservoir to a cylindric rotating tank of radius r. All the vessels are filled with water; there are holes of net cross-sectional area S_2 along the perimeter of the tank, which are open for the operating regime of the pump. The height of the tank from the free water surface of the reservoir is h (the height of the tank itself is small). An electric engine keeps the vessel rotating at the angular velocity ω. The water density is ρ, the air pressure — p_0, and the saturated vapour pressure — p_k. Assume the water flow to be laminar (neglect the energy of vortices) and neglect the friction. Inside the tank, there are metal blades, which make the water rotating together with the tank.

1) Calculate the pressure p_2 at the perimeter of the tank, when all the holes are closed (2 p).

2) From now on, all the holes are open. Find the velocity v_2 of the water jets with respect to the ground (2 p).

3) If the tank rotates too fast, the pump efficiency drops due to cavitation: the water starts "boiling" in some parts of the pump. Find the highest cavitation-free angular speed ω_{max} (3 pts).

4) If the power of the electric engine is P, what is the theoretical upper limit of the volume productivity of the pump μ_{max} (volume of the pumped water per unit time)?
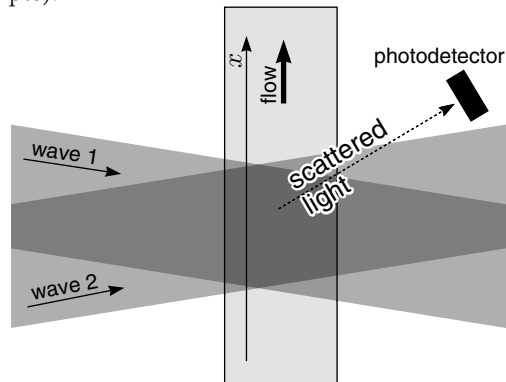
5. Anemometer (6 points)

Anemometer is a device measuring flow rate of a gas or a fluid. Let us look the construction of a simple laser-anemometer. In a rectangular pipe with thin glass walls flows a fluid (refractive index $n = 1{,}3$), which contains light dissipating particles. Two coherent plane waves with wavelength $\lambda = 515\,\text{nm}$ and angle $\alpha = 4°$ between their wave vectors, are incident on a plate so that *(a)* angle bisector of the angle between wave vectors is normal to one wall of the pipe and *(b)* pipe is parallel to the plane defined by wave vectors. Behind the pipe is a photodetector, that measures the frequency of changes in dissipated light intensity.

1) How long is the (spatial) period Δ of the interference pattern created along x-axis (see Figure; 2 pts)?

2) Let the oscillation frequency of the photometer signal be $\nu = 50\,\text{kHz}$. How large is the fluid's speed v? What can be said about the direction of the fluid flow (2 pts)?
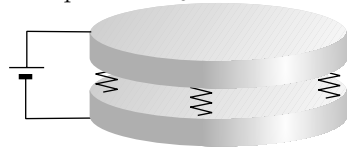
3) Let us consider a situation, when the wavelengths of the plane waves differ by $\delta\lambda = 4{,}4\,\text{fm}$ (1 fm$= 10^{-15}\,\text{m}$). What is the frequency of signal oscillations now (fluid's speed is the same as in previous section)? Is it possible to determine the flow direction with such a device (2 pts)?

6. Mechano-electrical oscillator (7 points)

Mechanical and electrical processes are sometimes strongly coupled. Very important examples are systems containing piezoelectric materials, e.g. quartz resonator. Here we investigate a somewhat simpler situation.

There are two metal plates with area S and mass m. One plate is situated atop of the other one. Plates are connected to each other with springs, whose total spring constant is k and what are made of insulator. The lower plate is mounted on a steady base. Equilibrium distance between the plates is X_0.

1) Let us assume that there is a small vertical shift x of the upper plate from its equilibrium position. Derive acceleration $\ddot{x}$ of x in terms of system parameters. What is the angular frequency ω_0 of the small vertical oscillations of the upper plate (1 pts)?

2) Plates are now connected to a constant high voltage source, so that they form a capacitor. Electrostatic force between the plates causes an additional shift of the upper plate. The equilibrium distance between the plates is now X_1. Derive expressions of the electrical attractive force F_e and voltage applied to the plates U in terms of X_0, X_1, S, m and k (2 pts).
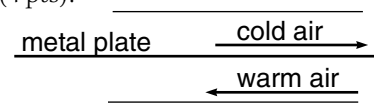
3) System is set to oscillate again, keeping voltage U constant. Let x stand still for the shift from the equilibrium position. Derive an expression for the acceleration $\ddot{x}$ of x in terms of X_0, X_1, S, m, k and shift x. What is the angular frequency ω_1 of upper plate's small vertical oscillations (2 pts)?

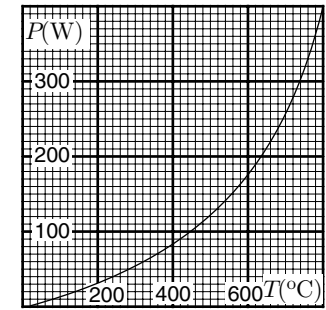**4)** Let us modify the situation of the previous question and connect an inductor with inductance L in series to the capacitor and voltage source. We describe the situation in terms of plate shift x and capacitor's charge q. Derive expressions for the accelerations $\ddot{x}$ and $\ddot{q}$ in terms of X_0, X_1, S, m, k, x and q. Which angular frequencies of harmonic oscillation are possible in the system? (2 pts)

7. Heat exchange (8 points)

1) Consider a simplified model of the air ventilation system of a house using a passive heat exchanger. The exchanger consists of a metal plate of length x and width y and thickness d dividing the air channel into two halves, one for incoming cold air, and another for outgoing warm air. Both channels have constant thickness h, air flow velocity is v see Figure. Thermal conductance of the metal is σ (the heat flux through a unit area of the plate per unit time, assuming that the temperature drops by one degree per unit thickness of the plate). Specific heat capacity of the air by constant pressure is c_p, air density is ρ (neglect its temperature dependance). You may assume that the air is turbulently mixed in the channel, so that the incoming and outgoing air temperatures T_{in} and T_{out} depend only on the coordinate x (the x-axes is taken parallel to the flow velocity), i.e. $T_{\text{in}} \equiv T_{\text{in}}(x)$ and $T_{\text{out}} \equiv T_{\text{out}}(x)$. Assuming that the inside and outside temperatures are T_0 and T_1, respectively, what is the temperature T_2 of the incoming air at the entrance to the room (4 pts)?
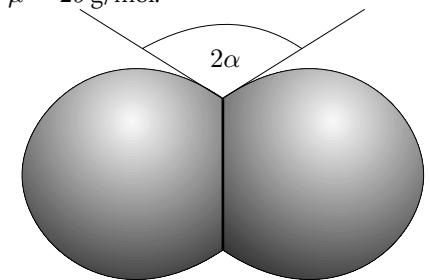

2) Attached is a plot of the heat exchange rate P of the wire of an electric heater as a function of temperature (assuming the room temperature is $T_0 = 20°\text{C}$). The operating temperature of the wire is $T_1 = 800°\text{C}$. The heater is switched off; find the time after which the temperature of the wire will drop down to $T_2 = 100°\text{C}$. The heat capacitance of the wire is $C = 10\,\text{J/K}$ (4 pts).

8. Balloon (8 points)

Find the mass of the balloon (including the gas inside it). Equipment: balloon (floats in air), digital scales, rope, measuring tape, rope fasteners, dynamometer, paper sheets for folding and rough angle measurements, 100-g weight, thread.

Remark: You may find it useful to know that if a rope is tied around a balloon so that the tension of the rope is T, the excess pressure inside the balloon is Δp, the angle between the tangents of the balloon envelope near the rope is 2α (average over the perimeter; see Figure), and the radius of the circular loop formed by the rope is R, then $\Delta p = T \tan\alpha/R^2$. The universal gas constant $R = 8.31\,\text{J/K·mol}$, the molar mass of air $\mu = 29\,\text{g/mol}$.

9. Mechanical black box (7 points)

There is something small inside the cylindrical "black box". Find the mass of it, as well as the friction coefficient between it and the inner surface of the box. Equipment: black box, ruler, a wooden plank, timer, scales.